Université Paris Dauphine
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Introduction to Time series
TD6 - Estimation

Delta method. Assume that (T_n) is a sequence of real random variables such that $\sqrt{n}(T_n - \theta)$ converges in law towards $\mathcal{N}(0, \sigma^2)$ as $n \to \infty$ and let $g \colon \mathbb{R} \to \mathbb{R}$ be a differentiable function with $g'(\theta) \neq 0$. Then, $\sqrt{n}(g(T_n) - g(\theta))$ converges in law towards $\mathcal{N}(0, \sigma^2 g'(\theta))$ as $n \to \infty$.

Empirical autocorrelation and Bartlett's formula. See Chapter 6 in D. Chafai et C. Lévy-Leduc's lecture notes.

Exercice 1 (Estimation of an AR(1)) Let (X_t) be an AR(1) process, i.e. $X_t = \phi X_{t-1} + Z_t$ where (Z_t) is a centered and very strong white noise having a moment of order 4. We suppose that $|\phi| < 1$.

1. Show that $\phi = \gamma_X(1)/\gamma_X(0)$.
2. Deduce a natural estimator for ϕ and give its asymptotic behaviour.

Exercice 2 (Estimation of an MA(1)) Let (X_t) be an MA(1) process, i.e. $X_t = \theta Z_{t-1} + Z_t$ where (Z_t) is a centered and very strong white noise having a moment of order 4. We assume that $|\theta| < 1$ and we will try to estimate it.

1. Show that θ can be expressed as a function of the correlation coefficient $\rho_X(1) = \gamma_X(1)/\gamma_X(0)$.
2. Give an estimator for $\rho_X(1)$ with its asymptotic behaviour.
3. Deduce an estimator of θ and give its asymptotic behaviour.

Exercice 3 (Estimation of the mean and confidence interval) Let $Y_t = \theta + X_t$, where (X_t) is an AR(1) defined by $X_t - \phi X_{t-1} = Z_t$, where $|\phi| < 1$ and the Z_t are independent and identically distributed with law $\mathcal{N}(0, \sigma^2)$. We seek to estimate θ from $Y_0, Y_1, \ldots, Y_{n-1}$. We note $\widehat{\theta}_n$ the empirical mean of $Y_0, Y_1, \ldots, Y_{n-1}$ defined by

$$\widehat{\theta}_n = \frac{1}{n} \sum_{i=0}^{n-1} Y_i.$$

1. Calculate $\lim_{n\to\infty} n \mathrm{var}(\widehat{\theta}_n)$ and give the expression of γ defined by
$$\sqrt{n}(\widehat{\theta}_n - \theta) \xrightarrow[n\to\infty]{\text{law}} \mathcal{N}(0, \gamma).$$
2. We choose $\phi = 0.6$ and $\sigma^2 = 2$. When we observe $n = 100$ values, we get $\widehat{\theta}_n = 0.271$. Build an interval asymptotic confidence at 95% for θ. Can we say that θ =0?
3. We propose another estimator of θ defined by $\widetilde{\theta}_n = (\mathbf{1}_n^\top \gamma_n^{-1} \mathbf{1}_n)^{-1} \mathbf{1}_n^\top \gamma_n^{-1} Y^{(n)}$ where $Y^{(n)} = (Y_0, Y_1, \ldots, Y_{n-1})^\top$, $\mathbf{1}_n = (1, \ldots, 1)^\top$ and γ_n is the matrix of covariance of $Y^{(n)}$. Justify the choice of this estimator;
4. Calculate $\lim_{n\to\infty} n \mathrm{Var}(\widetilde{\theta}_n)$. What do you conclude ?